Directorate of Distance and Continuing Education

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Tirunelveli-627 012, Tamil Nadu.

B.A. ECONOMICS

MATHEMATICS FOR ECONOMICS

JMEC32

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Mathematics for Economics

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Variables, Constants, Equations and its types – Uses and limitations of Mathematics in Economics- Functions of one or more variables – Linear function, Parabola, Rectangular Hyperbola- Exponential, Logarithmic, Power function and Homogenous Function – Applications in Economics

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UNIT III Applications of Matrix Algebra

Solving a system of Linear Equations – Cramer's Rule and Matrix Inverse Method Leontief's Input-Output Model – Open and Closed Model- Components, Uses, and Limitations- Hawkins – Simon Conditions for Viability of Input and Output Model (Problems).

UNIT IV Differentiation

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UNIT V Optimization (Single Variable)

Second Order Derivatives–Maximization and Minimization of a Function– Economic Applications – Output and Revenue Maximization -Cost Minimization – Profit Maximization under Perfect Competition, Monopoly, Discriminating Monopoly (Problems).

Textbooks

1 Mehta and Madnani (2019) Mathematics for Economists Sultan Chand and Sons

2 Edward T. Dowling,(2002) "Mathematical Methods for Business and Economics", Schaum's Outline Series, 3rd Edition, Mc Graw Hill

3 Renshaw Geoff, (2005) Maths for Economics, 3rd Edition Oxford University Press, Oxford

4 Carl P Simon & Lawrence E. Blume,"Mathematics for Economists", Published by W. W. Norton & Company,2010

5 Ian Jacques, "Mathematics for Economics and Business", Pearson, 2018.

UNIT I
INTRODUCTION

Variable

In any mathematical investigation, those quantities which are capable of taking different values in a particular analysis of the problem are called "variable". In other words, a variable is something whose magnitude can change something that can take different values. Usually variables are represented by the last letters of English alphabet: x, y, z.

Example:

In economics, the price per unit of the commodity, revenue, profit, cost, the quantity demanded In the short run market, income etc., are "variables".

Types of variables

a) Dependent and Independent variables

To find the area of a rectangle, the length and breadth of the rectangle are multiplied. The length and breadth can be of any value but its area is dependent on the values of the length and breadth of the rectangle. In this, length and breadth are called "Independent Variables" and the area is called "Dependent Variable".

b) Discrete variable and Continuous Variable

If the variable can take only a limited number of values within the interval of given values, it is called a "Discrete Variable", whereas if the variable can take any value within a given range of values, it is called a "Continuous Variable".

Constant

In any mathematical investigation those quantities which have the same value throughout the particular analysis of the problem are called "Constants". In other words, a constant is something whose magnitude does not change and is, therefore, the antithesis of a variable. In various economic phenomena, the numerical values of certain quantities remain constant. Usually constants are represented by the first letters of the English alphabet a,b,c, etc.

Examples:

1. Whatever may be the length of the sides of a triangle, the sum total of its angles is always 180 degrees. Hence, the sum total of angles of a triangle is ‘Constant’.
2. The income of household’s buys the commodity from the short run market is “Constant”.

Types of Constants

Broadly speaking constants are of two types, namely,

A) Absolute or Numerical Constants.

Absolute or Numerical Constants are those constants whose values are specific and permanently fixed. In other words, the constants which have the same value in all problems are called “Absolute Constant”.

Example:

The distance from Sivakasi to Madurai is 65 kms.

B) Arbitrary Constant or Parameters.

Arbitrary Constants or Parameters are those constants which have one value throughout one problem but may change from one problem to another problem. Usually Arbitrary Constants are represented by a, b, c…

Example:

$Y = mx$

Where ‘m’ is an arbitrary constant.

Equations

An equation is a statement of the equality of two algebraic expressions. The two equal expressions are called the “member” or “sides” of the equation. If the two sides of an equation are true for all values of the letters involved in the equation, then the equality is called “Identity” or “Identical Equation”.

An equation is a statement showing the equality two mathematical expressions. Each expression is called member or side of the equation.

Degree of an Equation

The highest power of the unknown variable is called the degree of the equation. The equation to the first power is called first degree of linear equation. It is second degree of quadratic equation if the power is 2 and cubic if the power is 3. If an equation has more than one term then, the equation will be called polynomials.

Types of equation

The following are the important types of equation on the basis of the highest power of the unknown variable.

1. Liner equations.
2. Quadratic equations.
3. Cubic equations.

Uses for Mathematics in Economics

Some of the most common uses for mathematics in economics include:

Analysis

Analysis is a key responsibility for an economics professional. Economic work often includes assessing information about economic performance, markets and other key economic data and extrapolating relevant information from the data. This allows individuals making economic decisions to do so while using information from various sources.

Math plays a large part in many forms of data analysis. This can include both the simple mathematics to perform tasks such as finding averages to advanced mathematics in the form of differential equations. Strong math skills in a diverse range of math capabilities can help an economist complete their analytical work more effectively.

Modelling

An economic model provides a visualization of key economic data. Using a model can make it easier for individuals to conceptualize or understand the state of an economic market. A model may also provide a new form for data that offers insights that the raw dataset does not.

As an economist, you are likely to use your math skills throughout the process of creating an economic model. Accurate math provides reliable data you can use in constructing a model, which can increase the value of the model provides upon completion.

Projection

Economic projections provide predictions of future economic behavior and patterns. Accurate projections are a valuable tool for economists, as it allows them to make decisions for future planning based on the state and behavior of the market in the future.

Math is an integral part of creating economic projections. It allows an economist to perform calculations on economic data, often using the principles of calculus to assess potential changes in the data over time. Developing your mathematical skills as an economist can help improve the accuracy of your calculations both by ensuring you complete them correctly and expanding the number of calculations and math principles you understand and may apply to your work.

Limitations of Mathematics in Economics

Mathematical economics can have limitations in a number of ways, including:

Subjective and unobservable elements

Economic processes involve subjective and unobservable elements that take place in the minds of economic agents. This makes it difficult to precisely define the terms being treated as quantities in a mathematical model.

Static models

Mathematical economics equations are static and can't describe the ever-changing marketplace. These equations attempt to depict a hypothetical "equilibrium" that can't be realized in a changing world.

Faulty models

Limitations can arise when a faulty economic model is set up and analyzed mathematically.

Inadequate use of mathematics

Limitations can arise when mathematics is used in an inadequate or incompetent manner.

Misleading results

Sophisticated mathematics can be used to cloak fundamentally misleading results and conclusions.

Functions

The word function was introduced into the mathematical language by Leobniz. Function indicates the association between two or more variables. If it is possible to calculate for a given X number, the corresponding y number, we will say that y is a function of X. symbolically y = f (X), this notation is due to Euler. The set of X values is called the domain of the function and y is called the range of the function.

The symbol whichcan assume various values is called variable. Since X and y are capable of taking many values they are called variables. The value of y depends on the values of X and so y is called 'dependent variable' and X the independent variable. We are interested in variables not for their own sake but for functional relationship between them.

Types of function

In mathematics, there are many functions. The following are the very important type of functions.

1. Algebraic function
2. Transcendental function
    a. Logarithmic functions
    b. Exponential functions
    c. Trigonometric functions

Algebraic functions

Algebraic functions are obtained through a finite number of algebraic operations like addition, subtraction, multiplication, division and through solving a finite number of algebraic equations. For example,

$$C = aq^2 + bq^2 - cq + d$$

The important algebraic functions can be classified as follows.

1. Mononomial and polynomial functions.
2. Linear and Non linear functions.
3. Quadratic, cubic and bi-quadratic functions.
4. Explicit and Implicit functions.

5. Homogenous and non-homogenous functions.
6. Rectangular hyperbolic functions.

1. Mononomial and polynomial functions

In a mononomial function, there is only one term in or example, $y = 2X$.

But, in polynomial function, there is more than one term. For example, $y=x^2+3x^2-5$.

2. Linear and Non-linear functions

Linear functions are otherwise called straight line function. The graph of the function will give a straight line or all the pairs of values lie on a straight line. For example, $y = a+bx$.

3. Quadratic, cubic and bi-quadratic functions.

In linear function the degree of the functions is one. But in quadratic function the degree is 2 and in the cubic function the degree is 3. On the other hand, in the bi-quadratic function, the degree if the functions is 4. In the graph of the quadratic function, there is only one bend against no bend in the linear equation. In the graph of the cubic function there are two bends.

4. Explicit and Implicit functions.

If the value of y is clearly seen to depend upon the value of X, (y=f(X)) it will be called an explicit function. For example, $y= aX^2 +bX + c$ is an explicit function.

But, in implicit function a functional relationship does not distinguish clearly between dependent and independent variables. For example,

$2x^2+ 3xy + y^2 = 0$ is an implicit function.

5. Homogenous and non-homogenous functions.

In homogenous functions, the sum of the powers of variables in each term in the right hand side is constant. On the other hand, in non-homogenous functions the sum of the powers of variables in each term is not constant. For example,

$Z= x^2=2xy+y^2$. it is a homogenous function as the sum of powers of variable in each term is equal to 2.

6. Rectangular hyperbolic functions.

It is a function in which, for every point on the curve represents the product of the distance of the point from two fixed perpendicular lines is a positive constant. For example $y=\frac{c}{x}$

Transcendental function

A function that is not algebraic is a transcendental function. Function like exponential, logarithmic and trigonometric functions are transcendental in nature.

1. Exponential functions

Let a > 0 and# 1. Then, the function ‘f’ defined by $f(x) = a^x$ is called an exponential function. The number ‘a’ is called the base of the function. The domain of ‘f’ is the set of all real numbers and the range of ‘f’ is the set of all positive numbers.

2. Logarithmic functions

The logarithm of a number to a given base is the power to which that base must be raised to give the number. Thus the logarithm if 100 to the base 10 is 2. The log of 9 to the base 3 is 2.The log of 8 to the base 2 is 3. If the base is 10, then the logarithm is called common logarithm and others are natural logarithms.

3. Trigonometric of circular functions

The circular functions applicable in economics are sin x and cos x. These functions are used in economics to explain cyclical functions. All circular functions are periodic in nature.

Number system

Tallying is the earliest method of counting. As man's knowledge developed, so also did the counting system. Having five fingers on each hand, man found it natural to count in groups of fives and tens. Form this came our present decimal system. Counting methods developed from fingers counting, to the abacus, to the present day electronic calculator. Different kinds of numbers such as natural numbers, int5egers, whole numbers, rational numbers, irrational numbers, real numbers and complex numbers were created to meet man's different needs.

a) **Natural numbers or Counting numbers**

The numbers which we use for counting purposes are called natural numbers. For example 1,2,3 and so on. They can be written down in succession without end. They, are generally, denoted by N.

b) **Whole numbers**

If '0' is included in the natural numbers, they form the set of whole numbers.

c) **Integers**

A number system having all the properties of N (Natural numbers) and also possessing the additive identity and additive inverses for all elements is called integer. We call the natural numbers as the positive integers and their inverses as the negative integers. The negative element of an element X is donted by –X. The identity element is called zero and is denoted by '0'. Integers are usually denoted by Z or I.

The set of integers Z = {……….-3,-2,-1,0,1,2,3,……}

d) **Rational numbers**

The division of an integer by another integer is called a fraction or a ratio or a rational number. Consider an equation of the form ax = b. this equation may or may not have solution in Z. for example, $5x = 20$ has solution in Z viz. $x = 4$. The equation $3x = 11$ has no solution in Z. The rational numbers system has all the essential properties of and also solution to all the equations of the form $ax = b$. The solution to the above equation is a rational number denoted by b/a where ‘b’ and ‘a’ are integers.

Rational numbers is positive or negative according ‘a’or ‘b’ is positive or negative. If both ‘a’ and ‘b’ are having the same sign, then the number is positive. If both are having different signs, then the number is negative.

e) Irrational numbers

Another set of numbers with which we are familiar the set of irrational numbers. An irrational number cannot be put in the form b/a, where a, b are integers. Such numbers were created through the efforts of Pythagoras almost 500 BC.

f) Real numbers

The rational numbers and the irrational number together called real numbers. Therefore every real numbers corresponds to some point on the number axis are corresponding to every point on the number axis there is some real number. For this reason the number axis is called the real line or real axis. The set of real.

g) Complex numbers

The equation $x^2 + 1 = 0$ has no solution in the real number system. There is no real number X such that $X^2 = -1$. Euler introduced the symbol ‘I’ to denote $\sqrt{-1}$ so that i = -1. Thus I is a solution of the above equation. He called ‘I’ an imaginary number. Let a and b be two real numbers. Any number if the form a+ ib is called a complex number.

UNIT II
MATRIX ALGEBRA AND DETERMINANTS

Meaning

In 1858, Cayley, the English Mathematician, invented the Theory of Matrices. An array of numbers in rectangular brackets is called ‘Matrix’. In other words, Matrix is a collection of vectors. Each Row and Column of the Matrix is a Vector.

Matrices:

Rectangular arrays of numbers that can represent a system of linear equations, linear transformations, and more.

Matrix Operations: Includes addition, subtraction, multiplication, and finding inverses.

Systems of Linear Equations:

A collection of linear equations involving the same set of variables.

Methods for solving systems include substitution, elimination, and using matrix operations like Gaussian elimination.

Determinants and Inverses:

Determinant: A scalar value that can be computed from the elements of a square matrix and provides important properties of the matrix (like whether it is invertible).

Inverse Matrix: A matrix that, when multiplied by the original matrix, yields the identity matrix. Not all matrices have inverses.

Eigenvalues and Eigenvectors:

Eigenvectors: Vectors that, when a linear transformation is applied, change only in scale (not direction).

Eigenvalues: Scalars that represent how much the eigenvector is scaled during the transformation.

Linear Transformations:

Functions that map vectors to other vectors in a linear manner. They can be represented by matrices.

Matrices

Determinants and matrices are fundamental concepts in linear algebra. They are instrumental in solving linear equations using Cramer’s rule, specifically for non-homogeneous equations in linear form. The determinant is applicable only for square matrices. If a matrix has a determinant of zero, it is referred to as a singular determinant. If the determinant is one, it is termed as unimodular. For a system of equations to have a unique solution, the determinant of the matrix should be nonzero, meaning it has to be non-singular. In this article, we will explore the definition, types, and properties of determinants and matrices, along with examples for better understanding.

Definition of Matrix

Matrices are essentially ordered rectangular arrays of numbers, utilized to represent linear equations. They consist of rows and columns, and various mathematical operations such as addition, subtraction, and multiplication can be performed on them. If a matrix has 'm' rows and 'n' columns, it is represented as an $m \times n$ matrix.

$$A = \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{bmatrix}$$

Now its determinant |A| is defined as

$$|A| = \begin{vmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{vmatrix}$$

$$= m\begin{vmatrix} r & s & t \\ v & w & x \\ z & a & b \end{vmatrix} - n\begin{vmatrix} q & s & t \\ u & w & x \\ y & a & b \end{vmatrix} + o\begin{vmatrix} q & r & t \\ u & v & x \\ y & z & b \end{vmatrix} - p\begin{vmatrix} q & r & s \\ u & v & w \\ y & z & a \end{vmatrix}$$

Various Types of Matrices

1) Row Matrix

A Matrix of order having only one row and n columns is called a “Row Vector” or “Row Matrix”. Row of the Matrix is a “Vector”.

Example:

$\left[a_{11}a_{12}a_{13}a_{14}\right]$.

2) Column Matrix

A Matrix of order having only one column and m rows is called a “Column Matrix” or “Column Vector”. Column of the Matrix is a “Vector”.

Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} 3\times1$$

3) Null Matrix or Column Matrix

If all the elements of a Matrix are zero, the Matrix is called a “Null Matrix” or “Coloumn Matrix”. Null Matrix is denoted by the symbol $\varnothing$ or 0.

Examples:

[0] Zero Matrix of order 1.

4)Sub- Matrix

Let ‘A’ be a given Matrix. By deleting a few rows and columns, we get a new Matrix called the “Sub- Matrix”. Clearly Matrix ‘A’ is also a Sub-Matrix of itself.

5) Equality of two Matrices or Equal Matrices

Two Matrix A and B are equal (A=B), if and only if they have the same order and that their corresponding elements are equal. That is, $a_{ij} = b_{ij}$, for all I,j.

Example:

$2\times 2 \;\; fA=\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$,then A= B.

6) Rectangular Matrix

If the number of rows of a Matrix is greatest than the number if its columns or the number of columns of a Matrix is greater than the number of its rows, then that Matrix is called a “Rectangular Matrix”, that is, m>n of n>m.

Examples:

$$\begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 3 \\ 4 & 2 \end{bmatrix} 4\times 2$$

7) Square Matrix

If the number of rows of a Matrix is equal to the number of its columns, then that Matrix is called a “Square Matrix”. In other words, a Matrix is square if it has the same number of rows and columns.

Examples:

$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is **a** 2×2 Square Matrix.

Matrices come in various types. Let's look at some examples of different types of matrices.

If A= $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a Null Matrix

$If\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ isa identity matrix

$If\ A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & 1 & 6 \\ -5 & 2 & 2 \end{bmatrix}$ is a symmetric matrix

$If\ A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is a diagonal matrix

$If\ A = \begin{bmatrix} 7 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 8 \end{bmatrix}$ is a upper diagonal matrix

$If\ A = \begin{bmatrix} 7 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 8 \end{bmatrix}$ is a lower diagonal matrix

Addition

Two Matrix A and B can be added if and only if they have the same order, the same number of rows and columns. That is number of column of Matrix A is equal to the number of column of B and number of Row of Matrix A is equal to the number of Row of B. That is, two Matrix of the same order are said to be "Conformable" for Addition. The sum of two Matrix of the same order is obtained by adding together corresponding elements of the two matrices.

Multiplication

Two Matrices A and B can be subtract if and only if they have the same order or dimension, the number of column of Matrix A is equal to the number of column of B and the number of Row of Matrix A is equal to the number of Row of B. in other words, Two Matrices of the same of order are said to be 'Conformable' for subtraction.

The subtraction of two matrices of the same order is obtained by subtracting corresponding elements. If A = $[a_{ij}]$ and B = $[b_{ij}]$, then C = A-B is the matrix having a general element of the form $c_{ij} = a_{ij} - b$

Examples:

If A = $\begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ and B = $\begin{bmatrix} 10 & 2 \\ 8 & 6 \end{bmatrix}$, find A-B and B – A.

Solution:

$$\text{A-B} = \begin{bmatrix} (1-10) & (5-2) \\ (6-8) & (7-6) \end{bmatrix} = \begin{bmatrix} -9 & -3 \\ -2 & -1 \end{bmatrix} 2\times 2$$

$$\text{B-A} = \begin{bmatrix} (10-1) & (9-3) \\ (8-6) & (6-7) \end{bmatrix} = \begin{bmatrix} 9 & -3 \\ 2 & -1 \end{bmatrix} 2\times 2$$

Transpose of a matrix

$\begin{bmatrix} (1-10) & (5-2) \\ (6-8) & (7-6) \end{bmatrix}$ The Transpose of a matrix 'A' is a New Matrix in which the rows and columns of Matrix A have been interchanged. That is, rows should be transferred or converted into columns and columns should be transferred or converted into rows. Transpose of a Matrix is denoted by A'or A^T.

Understanding the Inverse of a Matrix

The inverse of a matrix is typically defined for square matrices. For every m × n square matrix, an inverse matrix exists. If A is a square matrix, then A -1 is the inverse of matrix A and abides by the property:

AA -1 = A -1 A = I, where I is the Identity matrix.

Note that the determinant of the square matrix should not be zero.

Transpose of a Matrix

The transpose of a matrix is obtained by interchanging its rows and columns. If A is a matrix, then the transpose of the matrix is denoted by A T .

For instance, if we consider a 3×3 matrix, say A, then the transpose of A, i.e., A T is given by:

$$If\ A = \begin{bmatrix} 7 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 3 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

If the given square matrix is a symmetric matrix, then the matrix A should be equal to A T.

This implies that A = A T .

Determinant

The determinant of a square matrix can be defined in various ways.

The simplest way to calculate the determinant is by considering the elements of the top row and their corresponding minors. You start by taking the first element of the top row and multiplying it by its minor, then subtract the product of the second element and its minor. Continue this process of alternately adding and subtracting the product of each element of the top row with its respective minor until all elements of the top row have been covered.

For instance, let's consider a 4×4 matrix A.

$$A = \begin{bmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{bmatrix}$$

Now its determinant |A| is defined as

$$|A| = \begin{vmatrix} m & n & o & p \\ q & r & s & t \\ u & v & w & x \\ y & z & a & b \end{vmatrix}$$

$$= m\begin{vmatrix} r & s & t \\ v & w & x \\ z & a & b \end{vmatrix} - n\begin{vmatrix} q & s & t \\ u & w & x \\ y & a & b \end{vmatrix} + o\begin{vmatrix} q & r & t \\ u & v & x \\ y & z & b \end{vmatrix} - p\begin{vmatrix} q & r & s \\ u & v & w \\ y & z & a \end{vmatrix}$$

Properties of Determinants

Determinants which will help us in simplifying its evaluation by obtaining the maximum number of zeros in a row or a column. These properties are true for determinants of any order. However, we shall restrict ourselves to determinants of order 3 only.

Property 1

The value of the determinant remains unchanged if both rows and columns are interchanged.

= a1 (b2 c3 – b3 c2) – a2 (b1 c3 – b3 c1) + a3 (b1 c2 – b2 c1)

By interchanging the rows and columns of Δ, we get the determinant.

Expanding $\Delta 1$ along first column, we get,

$\Delta 1 = a1\ (b2\ c3 - c2\ b3) - a2\ (b1\ c3 - b3\ c1) + a3\ (b1\ c2 - b2\ c1)$

Hence $\Delta = \Delta 1$

Property 2:

If any two rows (or columns) of a determinant are interchanged, then the sign of determinant changes.

Expanding along first row, we get,

$\Delta = a1\ (b2\ c3 - b3\ c2) - a2\ (b1\ c3 - b3\ c1) + a3\ (b1\ c2 - b2\ c1)$

Interchanging first and third rows, the new determinant obtained

Expanding along third row, we get,

$\Delta 1 = a1\ (c2\ b3 - b2\ c3) - a2\ (c1\ b3 - c3\ b1) + a3\ (b2\ c1 - b1\ c2)$

$= -\ [a1\ (b2\ c3 - b3\ c2) - a2\ (b1\ c3 - b3\ c1) + a3\ (b1\ c2 - b2\ c1)]$

Clearly $\Delta 1 = -\Delta$

Similarly, we can verify the result by interchanging any two columns.

Property 3:

If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then the value of the determinant is zero.

Proof: If we interchange the identical rows (or columns) of the determinant Δ, then Δ does not change. However, by Property 2, it follows that Δ has changed its sign, therefore $\Delta = -\Delta$ or $\Delta = 0$.

Property 4:

If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Property 5:

If some or all elements of a row or column of a determinant are expressed as the sum of two (or more) terms, then the determinant can be expressed as the sum of two (or more) determinants.

Property 6:

If the equimultiples of corresponding elements of other rows (or columns) are added to every element of any row or column of a determinant, then the value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation Ri → Ri + k Rj or Ci → Ci + k Cj .

Properties of Matrix Inverse

Inverse of 2×3 Matrix

A matrix must be non-singular in order to have an inverse matrix. For it to be non-singular, the determinant of a matrix must be non-zero. And the determinant only exists for square matrices. This indicates that where m x n, the inverse of matrices of order m ≠ n will not exist. As a result, we are unable to compute the inverse of the 2×3 matrix.

Important Points on Inverse of a Matrix

The following points will help you comprehend the concept of matrix inverse more fully.

- The inverse of a square matrix is unique.
- If A and B are two same-order invertible matrices, then (AB)-1 = B-1 A-1
- Only if the determinant of a square matrix A is non-zero, i.e. |A| ≠ 0, does the inverse exist.
- When the elements of a row or column are multiplied by the elements of any other row or column, their sum is zero.
- The product of the determinants of two matrices is the product of the determinants of the two individual matrices. |AB| = |A||B|.

UNIT III

APPLICATION OF MATRIX ALGEBRA

Solving a system of Linear Equations:

1. Cramer's rule for solving system of linear equations

Here, one has to find out $\Delta, \Delta_x, \Delta_y, \Delta_z$. Δ is the determinant of the given matrix.
Δ_x is the determinant of matrix replacing the first column by the constant matrix
Δ_y is the determinant of matrix replacing the second column by the constant matrix
Δ_z is the determinant of matrix replacing the third column by the constant matrix

Solve the following equations by using Cramer's rule:

$$2x+3y+4z=29$$
$$3x+2y+5z=32$$
$$4x+3y+2z=25$$

$$x=\frac{\Delta_x}{\Delta},\ y=\frac{\Delta_y}{\Delta},\ z=\frac{\Delta_z}{\Delta}$$

Solution:

$$\Delta=\begin{vmatrix}2 & 3 & 4\\ 3 & 2 & 5\\ 4 & 3 & 2\end{vmatrix}$$
$$=2(2\times 2)-(5\times 3)-3(3\times 2)-(5\times 4)+4(3\times 3)-(2\times 4)$$
$$=2(4-15)-3(6-20)+4(9-8)$$
$$=2(-11)-3(-14)+4(1)$$
$$=-22+42+4$$
$$\Delta=24$$

$$\Delta_x=\begin{vmatrix}29 & 3 & 4\\ 32 & 2 & 5\\ 25 & 3 & 2\end{vmatrix}$$
$$=29(2\times 2)-(5\times 3)-3(32\times 2)-(5\times 25)+4(32\times 3)-(2\times 25)$$
$$=29(4-15)-3(64-125)+4(96-50)$$
$$=29(-11)-3(-61)+4(46)$$
$$=-319+183+184$$
$$=-319+367$$
$$\Delta_x=48$$

$$\Delta_y = \begin{vmatrix} 2 & 29 & 4 \\ 3 & 32 & 5 \\ 4 & 25 & 2 \end{vmatrix}$$

$= 2(32\times 2)-(5\times 25)-29(3\times 2)-(5\times 4)+4(3\times 25)-(32\times 4)$

$=2(64-125)-29(6-20)+4(150-128)$

$= 2(-61)-29(-14)+4(-53)$

$=-122+406+312$

$=-334+406$

$\Delta_y = 72$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 29 \\ 3 & 2 & 32 \\ 4 & 3 & 25 \end{vmatrix}$$

$= 2(2\times 25)-(32\times 3)-3(3\times 25)-(32\times 4)+29(3\times 3)-(2\times 4)$

$=2(50-96)-3(75-128)+29(9-8)$

$= 2(-46)-3(-53)+29(1)$

$=-92+159+29$

$=-92+188$

$\Delta_z = 96$

$x = \frac{\Delta_x}{\Delta} = \frac{48}{24} = 2$

$y = \frac{\Delta_y}{\Delta} = \frac{72}{24} = 3$

$z = \frac{\Delta_z}{\Delta} = \frac{96}{24} = 4$

$x = 2,\ y = 3,\ z = 4$

2. Inverse of Matrix

Solve the following simultaneous equations for x and y

$3x+2y=5$

$7x+3y=10$

Solution:

$3x+2y=5$

$7x+3y=10$

$$\begin{bmatrix} 3 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$Apply\ A^{-1}\times B$

$A^{-1} = \frac{AdjA}{|A|}$

$$|A| = \begin{bmatrix} 3 & 2 \\ 7 & 3 \end{bmatrix} = 9 - 14 = -5$$

$$Adj\ A = \begin{bmatrix} 3 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{-5} & \frac{-2}{-5} \\ \frac{-7}{-5} & \frac{3}{-5} \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$A^{-1} = \begin{bmatrix} \frac{3}{-5} & \frac{-2}{-5} \\ \frac{-7}{-5} & \frac{3}{-5} \end{bmatrix} \times \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} \frac{3}{-5} \times 5 & \frac{-2}{-5} \times 10 \\ \frac{-7}{-5} \times 5 & \frac{3}{-5} \times 10 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 + 4 \\ 7 - 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x = 1, y = 1$$

Simultaneous linear equations

Solve the following simultaneous equations for x, y and z:

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 9z = 36$$

The given set of equations can be written as AX=B

Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$

$$Apply\ A^{-1} \times B$$

$$A^{-1} = \frac{AdjA}{|A|}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + 1(4-2)$$

$$= 1(6) - 1(6) + 1(2)$$

$$= 6 - 6 + 2$$

$$|A| = 2$$

Co-factor of A

$$A_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 18 - 12 = 6$$

$$A_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9-3) = -6$$

$$A_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 4 - 2 = 2$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9-4) = -5$$

$$A_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = (9-1) = 8$$

$$A_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$A_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$A_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$A_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$$

$$Co-Factor\,of\;A = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$Adj\;A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$Apply\ X = A^{-1} \times B$$

$$= \begin{bmatrix} \frac{6}{2} & \frac{-5}{2} & \frac{1}{2} \\ \frac{-6}{2} & \frac{8}{2} & \frac{-3}{2} \\ \frac{1}{2} & \frac{-2}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$

Ans = $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3. Leontief's Input-Output Analysis

Input and output models

Introduction

The Input-Output Analysis was first propounded by Prof.Wassily W. Leontief of Harvard University in 1951 for which he was honored with the 1973 Nobel Prize. Therefore, he is called as the "Father" of the Input-Output Analysis. His model is based on the earlier works done in this field by the French Economist, Francois Quesnay in his "Tableau Economic", Karl Marx in his "Reproduction Schema" and Leon Walras in his "General Equilibrium.

Input-output is a novel technique invented by Professor Wassily W. Leontief in 1951. It is used to analyse inter-industry relationship in order to understand the inter-dependencies and complexities of the economy and thus the conditions for maintaining equilibrium between supply and demand.

Thus it is a technique to explain the general equilibrium of the economy. It is also known as "inter-industry analysis". Before analysing the input-output method, let us understand the meaning of the terms, “input” and “output”. According to Professor J.R. Hicks, an input is “something which is bought for the enterprise” while an output is “something which is sold by it.”

An input is obtained but an output is produced. Thus input represents the expenditure of the firm, and output its receipts. The sum of the money values of inputs is the total cost of a firm and the sum of the money values of the output is its total revenue.

Limitations

1. The I-O analysis assumes fixed input-output coefficients. I ignores the possibility of factor substitution.
2. This analysis needs more time and money.
3. The assumption of constant returns to scale is unrealistic.
4. This analysis assumes the absence of external economies and diseconomies of production. It does not hold good in practice.
5. The assumption of no change in technology is unrealistic.
6. Errors in forecasting final demand may have severe consequences.
7. It deals only with productive sectors and neglects the essential unproductive sectors.
8. There are also practical difficulties in the formation of I-O tables and technical coefficients.
9. It is very difficult to understand and make use of it for practical purpose.

Importance

1. The I-O analysis helps to know the extent of inter-relationship between different sectors of the economy.
2. It helps a producer to know the quantity to be produced and sold by him and by other fims.

3. This analysis is very much useful in national income accounting.

4. The effect of war, trade cycles and prolonged strikes can be understood with the help of I–*O* analysis.

5. A dynamic I-O model is more appropriate and important in economic planning.

The input-output analysis tells us that there are industrial interrelationships and inter-dependen-cies in the economic system as a whole. The inputs of one industry are the outputs of another industry and vice versa, so that ultimately their mutual relationships lead to equilibrium between supply and demand in the economy as a whole.

Coal is an input for steel industry and steel is an input for coal industry, though both are the outputs of their respective industries. A major part of economic activity consists in producing intermediate goods (inputs) for further use in producing final goods (outputs).

There are flows of goods in "whirlpools and cross currents" between different industries. The supply side consists of large inter-industry flows of intermediate products and the demand side of the final goods. In essence, the input-output analysis implies that in equilibrium, the money value of aggregate output of the whole economy must equal the sum of the money values of inter-industry inputs and the sum of the money values of inter-industry outputs.

1. Main Features:

The input-output analysis is the finest variant of general equilibrium. As such, it has three main elements; Firstly, the input-output analysis concentrates on an economy which is in equilibrium. Sec-ondly, it does not concern itself with the demand analysis. It deals exclusively with technical problems of production. Lastly, it is based on empirical investigation. The input-output analysis consists of two parts: the construction of the input-output table and the use of input-output model.

2. The Static Input-Output Model:

Assumptions:

This analysis is based on the following assumptions:

(i) The whole economy is divided into two sectors—“inter-industry sectors” and “final-demand sectors,” both being capable of sub-sectoral division.

(ii) The total output of any inter-industry sector is generally capable of being used as inputs by other inter-industry sectors, by itself and by final demand sectors.

(iii) No two products are produced jointly. Each industry produces only one homogeneous product.

(iv) Prices, consumer demands and factor supplies are given.

(v) There are constant returns to scale.

(vi) There are no external economies and diseconomies of production.

(vii) The combinations of inputs are employed in rigidly fixed proportions. The inputs remain in constant proportion to the level of output. It implies that there is no substitution between different materials and no technological progress. There are fixed input coefficients of production.

The input-output model relates to the economy as a whole in a particular year. It shows the values of the flows of goods and services between different productive sectors especially inter-industry flows.

Explanation:

For understanding, a three-sector economy is taken in which there are two inter-industry sec-tors, agriculture and industry, and one final demand sector.

Table 1 provides a simplified picture of such economy in which the total output of the industrial, agricultural and household sectors is set in rows (to be read horizontally) and has been divided into the agricultural, industrial and final demand sectors. The inputs of these sectors are set in columns. The first row total shows that altogether the agricultural output is valued at Rs. 300 crores per year.

Table 1 : Input-Output Table

(In value terms) (Rs. Crores)

	Sectors	1 Inputs to Agriculture	2 Inputs to Industry	3 Final Demand	Total Output or Total Revenue
		Purchasing Sectors			
Selling Sectors	Agriculture	50	150	100	300
	Industry	100	250	150	500
	Value added*	150	100	0	250
	Total input or Total Cost	300	500	250	1050

Of this total, Rs. 100 crores go directly to final consumption (demand), that is, household and government, as shown in the third column of the first row. The remaining output from agriculture goes as inputs: 50 to itself and 150 to industry. Similarly, the second row shows the distribution of total output of the industrial sector valued at Rs. 500 crores per year. Columns 1, 2 and 3 show that 100 units of manufactured goods go as inputs to agriculture, 250 to industry itself and 150 for final con-sumption to the household sector.

Let us take the columns (to be read downwards). The first column describes the input or cost structure of the agricultural industry. Agricultural output valued at Rs. 300 crores is produced with the use of agricultural goods worth Rs. 50, manufactured goods worth Rs. 100 and labour or/and manage-ment services valued at Rs. 150. To put it differently, it costs Rs. 300 crores to get revenue of Rs. 300 crores from the agricultural sector. Similarly, the second column explains the input structure of the industrial sector (i.e., 150 + 250 + 100 = 500).

Thus "a column gives one point on the production function of the corresponding industry." The 'final demand' column shows what is available for consumption and government expenditure. The third row corresponding to this column has been shown as zero. This means that the household sector is simply a spending (consuming) sector that does not sell anything to itself. In other words, labour is not directly consumed.

There are two types of relationships which indicate and determine the manner in which an economy behaves and assumes a certain pattern of flows of resources.

They are:

(a) The internal stability or balance of each sector of the economy, and

(b) The external stability of each sector or inter-sectoral relationships. Professor Leontief calls them the "fundamental relationships of balance and structure." When expressed mathematically they are known as the "balance equations' and the "struc-tural equations".

If the total output of say X. of the 'ith' industry is divided into various numbers of industries 1, 2, 3, n, then we have the balance equation:

X1 = xi1 + xi2 + xi3 + xin…… + D1

and if the amount say У. absorbed by the "outside sector" is also taken into consideration, the balance equation of the ith industry becomes

It is to be noted that Yi stands for the sum of the flows of the products of the ith industry to consump-tion, investment and exports net of imports, etc. It is also called the "final bill of goods" which it is the function of the output to fill. The balance equation shows the conditions of equilibrium between de-mand and supply. It shows the flows of outputs and inputs to and from one industry to other industries and vice versa.

Since x12 stands for the amount absorbed by industry 2 of the ith industry, it follows that xij stands for the amount absorbed by the ith industry of jth industry.

The "technical coefficient" or "input coefficient" of the ith industry is denoted by:

aij = xij/Xj

wherexij is the flow from industry i to industry j, Xj is the total output of industry aij and aij, as already noted above, is a constant, called "technical coefficient" or "flow coefficient" in the ith industry. The technical coefficient shows the number of units of one industry's output that are required to produce one unit to another industry's output.

Equation (3) is called a "structural equation." The structural equation tells us that the output of one industry is absorbed by all industries so that the flow structure of the entire economy is revealed. A number of structural equations give a summary description of the economy's existing technological conditions.

Using equation (3) to calculate the aij for our example of the two-sector input-output Table 1, we get the following technology matrix.

Table 2: Technology Coefficient Matrix A

	Agriculture	*Industry*
Agriculture	50/300= .17	150/500= .30
Industry	100/300= .33	250/500= .50

These input coefficients have been arrived at by dividing each item in the first column of Table 1 by first row total, and each item in the second column by the second row, and so on. Each column of the technological matrix reveals how much agricultural and industrial sectors require from each other to produce a rupee's worth of output. The first column shows that a rupee's worth of agricultural output requires inputs worth 33 paise from industries and worth 17 paise from agriculture itself.

Hawkins - Simon condition open and closed models

1. The Hawkins – Simon conditions

Hawkins – Simon conditions ensure the viability of the system.

• If B is the technology matrix Hawkins – Simon conditions are

i. the main diagonal elements in I – B must be positive and

ii. |I – B| must be positive.

Types

1.**The Open Model**

If, besides the n industries, the model contains an "open" sector (say,households) which exogenously determines a final demand (non-inputdemand) for the product of each industry and which supplies a primaryinput (say, labour service) not produced by the n industries themselves,then the model is an open one

2. **The Closed Model**

If the exogenous sector of the open input-output model is absorbed intothe system as just another industry, the model will become a closed one. Insuch a model, final demand and primary input do not appear; in their placewill be the input requirements and the output of the newly conceivedindustry. All goods will now be intermediate in nature, because everythingis produced only for the sake of satisfying the input requirements of the (n + 1) sectors in the model.

The Input-Output Model is an analytical framework primarily used in economics to examine the flow of goods and services within an economy or between different sectors. It helps identify interdependencies between industries and the relationships between inputs (resources used) and outputs (goods or services produced). Here are the merits and demerits of this model:

Merits of the Input-Output Model

Comprehensive Analysis:

It provides a detailed representation of economic activities by showing the interrelationships among industries and sectors.

Identifying Key Sectors:

The model helps pinpoint industries that are vital to the economy, facilitating targeted policies for growth.

Economic Impact Assessment:

Useful for evaluating the effects of changes in one sector (e.g., increased production, policy changes) on other sectors and the overall economy.

Planning and Forecasting:

Governments and organizations use the model for resource allocation, economic planning, and predicting economic trends.

Quantitative Decision-Making:

The framework offers measurable data that assist in informed decision-making.

Policy Evaluation:

It aids in assessing the economic impact of new projects, policies, or investments.

Demerits of the Input-Output Model

Static Nature:

The model assumes fixed relationships between inputs and outputs, which does not reflect dynamic changes like technological advancement.

Linear Assumptions:

It relies on linear relationships, ignoring economies of scale, substitution effects, or diminishing returns.

Data Intensity:

Developing and maintaining accurate input-output tables requires extensive and reliable data, which may not always be available.

Lack of Flexibility:

The model struggles to account for changes in market conditions, such as supply shocks, price fluctuations, or global trade disruptions.

Over-Simplification:

By aggregating data, the model may overlook regional, cultural, or firm-specific differences.

Resource-Dependent:

High costs and time are required to build, update, and analyze the complex matrix of input-output tables.

Limited to Short-Term Analysis:

Long-term economic shifts, like structural changes in industries, are not well-captured by the model.

In summary, while the Input-Output Model is a powerful tool for understanding and planning economic activities, its limitations arise from its static and data-intensive nature. For dynamic or rapidly changing economies, it should be complemented with other models or methodologies.

UNIT IV

DIFFERENTATION

CALCULUS

MEANING: Calculus is the branch of mathematics of change, motion and growth in related variables. It is the science of fluctuations. Therefore, in Economics, Calculus has a role to play when we consider how the sales is affected when there is change in the price or how the total cost, price etc., are affected when there is change in output and so on.

Branches of Calculus

The branches of Calculus are:

A) Differential Calculus and

B) Integral Calculus.

A) Differential Calculus

Meaning: Differentiation is the process of finding the rate at which a variable quantity is changing. To express the rate of change in any function, we have the concept of the 'Derivative' which involves small change in the dependent variables with reference to a small change in independent variables. The problem is to find a function derived from the given relationship between the two variables so as to express the idea of change. This derived function is called the "Derivative" of a given function. The process of obtaining the derivative is called "Differentiation". When a function has a Derivative, it is said to be Differentiable.

Thus,

1) The Marginal Cost is the rate of change of total cost with change in quantity produced.
2) The Marginal Revenue is the rate of change of total revenue with change in quantity produced.
3) The Marginal Utility is the rate of change of total utility with change in quantity consumed.
4) The Marginal Productivity is the rate of change of total productivity with change in factors of production.

Differential Calculus of One Variable

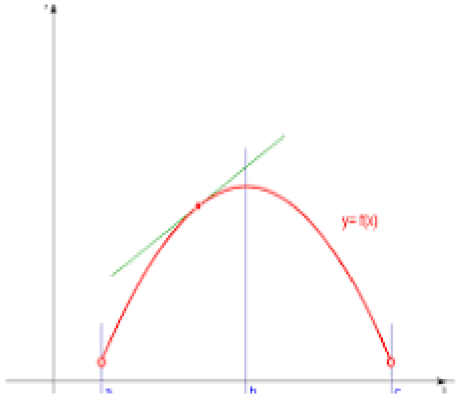
Let 'Y' be a continuous and single valued function of 'x'. Then an increase in the value of x will produce a corresponding increase or decrease in the value of Y. That is a function is differentiable at a point only if it is a single valued function of a continuous variable at that point. In other words, the derivative of y with respect to x exists only when the function is a single valued function of a continuous variable.

Differentiability of a Function

The process of finding derivatives of a function is called differentiation in calculus. A derivative is the rate of change of a function with respect to another quantity.

Slope of a Curve

The slope of a curve is the steepness of the curve at a given point, and is measured by the slope of the tangent line that touches the curve at that point.



Differential calculus defines the slope of a plane curve at a point as the slope of its tangent line at that point. When the curve is approximated by a series of points, the slope of the curve may be approximated by the slope of the secant line between two nearby points.

Increasing and Decreasing Functions

To determine whether a function is increasing or decreasing using derivatives, you can check the sign of the function's first derivative:

Increasing: If the first derivative is positive, then the function is increasing.

Decreasing: If the first derivative is negative, then the function is decreasing.

Constant: If the first derivative is zero, then the function is constant.

Rules- Exponential and Logarithmic Functions-Implicit Differentiation

1. Power Rule

$$\begin{aligned} &If\ \ y = x^n \\ &\frac{dy}{dx} = nx^{n-1} \end{aligned}$$

Example

$$\begin{aligned} &y = x^3 \\ &\frac{dy}{dx} = 3x^{3-1} \\ &= 3x^2 \end{aligned}$$

2. Constant Rule

$$\begin{aligned} &y = c \\ &\frac{dy}{dx} = 0 \end{aligned}$$

Example

$$\begin{aligned} &y = 10 \\ &\frac{dy}{dx} = 0 \end{aligned}$$

3. Sum or Difference Rule

$$\begin{aligned} &y = u - v \\ &\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \end{aligned}$$

Example

$$\begin{aligned} &y = 10x^4 - 20x^2 + 5x \\ &\frac{dy}{dx} = 4 \times 10x^{4-1} - 2 \times 20x^{2-1} + 5 \\ &\frac{dy}{dx} = 40x^3 - 40x + 5 \end{aligned}$$

4. Product Rule

If the function is a product of two term, then $y = u \times v$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

Example

$y=(2x^2+5x)(x^2-5x)$

$$u=2x^2+5x \quad \mathrm{v}=x^2-5x$$

$$\frac{du}{dx}=4x+5 \quad \frac{dv}{dx}=2x-5$$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

$$\frac{dy}{dx}=(x^2-5x)(4x+5)+(2x^2+5x)(2x-5)$$

$$=4x^3+5x^2-20\,\mathrm{x}^2-25x+4x^3-10x^2+10x^2-25x$$

$$\frac{dy}{dx}=8x^3-15x^2-50x$$

5. Division Rule

If the function is a product of two term, then $y=\frac{u}{v}$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Example

$y=\frac{7x-2}{5x+3}$

$$u=7x-2 \quad \mathrm{v}=5x+3$$

$$\frac{du}{dx}=7 \quad \frac{dv}{dx}=5$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{5x+3(7)-(7\,\mathrm{x}-2)5}{(5x+3)^2}$$

$$= \frac{35x + 21 - 35x + 10}{25x^2 + 30x + 9}$$

$$\frac{dy}{dx} = \frac{31}{25x^2 + 30x + 9}$$

6. Chain Rule

If the function is a function, then $\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dv}{dx}$

Example

$$y = (2 + 3x^2)^2$$

$$y = u^2 \qquad u = 2 + 3x^2$$

$$\frac{dy}{du} = 2u \qquad \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = 2u(6x)$$

$$= 2(2 + 3x^2)6x$$

$$= 12x(2 + 3x^2)$$

$$\frac{dy}{dx} = 24x + 36x^3$$

7. Logarithmic Rule

$$i)\ y = \log x$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$ii)\ y = \log u$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example

$$If\ \ y = \log(5x^2 + 2)$$

$$\mathrm{u} = 5x^2 + 2$$

$$\frac{du}{dx} = \frac{1}{u} \qquad \frac{du}{dx} = 10x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u}(10x)$$

$$= \frac{1}{5x^2+2}(10x)$$

$$= \frac{10x}{5x^2+2}$$

8. Exponential Rule

$$i)\ y = e^x$$

$$\frac{dy}{dx} = e^x$$

$$ii)\ \ y = e^{ax}$$

$$\frac{dy}{dx} = ae^{ax}$$

$$iii)\ y = e^u$$

$$\frac{dy}{dx} = \frac{du}{dx}.e^u$$

Example

$$If\ \ y = e^{5x+3}\ \ find\ {}^{dy}\!/_{dx}$$

$$u = 5x+3$$

$$\frac{du}{dx} = 5$$

$$\frac{dy}{dx} = \frac{du}{dx}.e^u$$

$$\frac{dy}{dx} = 5.e^{5x+3}$$

9. Implicit Function

$$x^2 + y^2 = 0$$

$$2x + 2y\frac{dy}{dx}$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Economic Applications: Marginal and Elasticity Concepts

1. The demand function is $P = 30 - Q$ and the cost function is $20 + 4Q$. Find the equilibrium and price of the firm using MC=MR techniques.

$$R = P \times Q$$
$$R = (30 - Q)Q$$
$$R = 30Q - Q^2$$

$$MR = \frac{dR}{dQ} = 30 - 2Q$$
$$MC = \frac{dC}{dQ} = 4$$

At equilibrium MC=MR

$$30 - 2Q = 4$$
$$-2Q = 4 - 30$$
$$-2Q = -26$$
$$Q = \frac{26}{2}$$
$$Q = 13$$

$$P = 30 - Q$$
$$= 30 - 13$$
$$= 17$$

Relationship between AR, MR, and Price Elasticity of Demand

The relationship between Average Revenue (AR), Marginal Revenue (MR), and Price Elasticity of Demand (PED) is critical in understanding how changes in price affect revenue and output decisions in economics. Here's a breakdown of this relationship:

1. Average Revenue (AR) and Price Elasticity of Demand

AR is essentially the price per unit sold.

The Price Elasticity of Demand (PED) measures the responsiveness of quantity demanded to a change in price.

The nature of PED directly affects AR:

Elastic demand (PED > 1): A decrease in price increases total revenue (and vice versa).

Inelastic demand (PED < 1): A decrease in price decreases total revenue (and vice versa).

Unitary elastic demand (PED = 1): Changes in price do not affect total revenue.

2. Marginal Revenue (MR) and Price Elasticity of Demand

MR is the additional revenue gained from selling one more unit of output.

The relationship between MR and PED is given by the formula:

$$MR - P(1+\frac{1}{PED})$$

Where P is the price and PED is the price elasticity of demand.

Key insights:

Elastic demand (PED > 1): $MR>0$ Increasing output leads to an increase in total revenue.

Inelastic demand (PED < 1): $MR<0$. Increasing output decreases total revenue.

Unitary elastic demand (PED = 1): $MR=0$. Total revenue is maximized, and further changes in output will not increase it.

3. AR and MR Relationship

AR represents the price, and MR is always less than AR under imperfect competition (due to the downward-sloping demand curve).

The MR curve lies below the AR curve in such markets. The gap between AR and MR depends on the elasticity of demand:

When demand is elastic (PED > 1), the gap between AR and MR is smaller.

When demand is inelastic (PED < 1), the gap between AR and MR widens.

Summary of the Relationships

The PED determines whether AR (price) increases or decreases with a change in quantity.

MR is positive when demand is elastic, zero at unitary elasticity, and negative when demand is inelastic.

A firm maximizes total revenue when PED = 1 (MR = 0), and at this point, AR is neither rising nor falling.

This understanding helps firms decide optimal pricing and output strategies to align with their revenue or profit-maximizing goals.

Relationship between Average and Marginal Cost.

The relationship between Average Cost (AC) and Marginal Cost (MC) is an important concept in economics, particularly in the study of production and cost analysis. Here's a detailed explanation:

1. Definitions

Average Cost (AC): The cost per unit of output, calculated as:

$$AC = \frac{Total\,Cost}{Quantity(Q)}$$

Marginal Cost (MC): The additional cost incurred when producing one more unit of output, calculated as:

$$MC = \frac{\Delta Total\,Cost}{\Delta Quantity(Q)}$$

2. Relationship between AC and MC

The relationship between AC and MC is based on the behaviour of these two costs as output changes:

When MC < AC:

The AC is decreasing. This is because the cost of producing an additional unit (MC) is lower than the current average, pulling the average down.

When MC > AC:

The AC is increasing. This is because the cost of producing an additional unit (MC) is higher than the current average, pulling the average up.

When MC = AC:

The AC is at its minimum point. This is the point where the average cost neither increases nor decreases, indicating the most efficient scale of production.

3. Graphical Representation

The AC curve is typically U-shaped, reflecting economies and diseconomies of scale.

The MC curve usually cuts the AC curve at its lowest point. This intersection point is crucial because it indicates the output level at which the cost per unit is minimized.

Elasticity of Demand

$$e = \frac{The\ proportionate\ Change\ in\ Quantity\ demanded}{The\ proportionate\ Change\ in\ \mathrm{Pr}\,ice}$$

$$e = \frac{\Delta Q}{\Delta P} \times \frac{P}{Q}$$

$$e = \frac{dQ}{dP} \times \frac{P}{Q}$$

Find the elasticity $Qd = 5P^2 - 2P$. Find the elasticity at P=5.

$$\frac{dQ}{dP} = 10P - 2$$

$$e = \frac{dQ}{dP} \times \frac{P}{Q}$$

$$= 10P - 2 \times \frac{P}{Q}$$

$$When\ P = 5 \quad Q = 5P^2 - 2P$$

$$= 10P - 2 \times \frac{P}{Q}$$

$$= 10(5) - 2 \times \frac{5}{5(5)^2 - 2(5)}$$

$$= \frac{(50-2)5}{5(25)-10}$$

$$= \frac{48(5)}{125-10}$$

$$= \frac{240}{115}$$

$$e = 2.01$$

UNIT V

OPTIMIZATION (SINGLE VARIABLE)

In calculus, the process of finding the maximum or minimum value of a function is called optimization.

Second Order Derivatives

The first-order derivative is the derivative of a function determined by one variable or the derivative of a dependent variable with respect to an independent variable. The second-order derivative is a derivative at a point chosen by two variables.

To understand what a second-order derivative is, we must first understand what a derivative is. A derivative gives you the slope of a function at any point. A second-order derivative is a function's derivative of its derivative. The first-order derivative is used to create it. So we get a function's derivative first, then draw the derivative of the first derivative. A first-order derivative is denoted by f'(x) or dy/dx, whereas a second-order derivative is denoted by f''(x) or d^2y/dx^2.

Maximization and Minimization of a Function

The function to maximize (minimize) is called the objective function. The maximum value (or minimum) of the objective function is in the margins of the feasible area delimited by the restrictions of the problem. This value is called the ideal value.

I- Revenue Function

Given the demand function P=40-2x. Find the output and price when the revenue is maximum**.**

Total Revenue = $P \times X$

$$R = (40 - 2x)x$$

$$R = 40x - 2x^2$$

$$\frac{dR}{dx} = 40 - 4x$$

First order condition for maximization $\frac{dR}{dx} = 0$

$$40 - 4x = 0$$

$$-4x = -40$$

$$4x = 40\,,\; x = 10$$

First order condition for maximization $\frac{d^2R}{dx^2} < 0$

$$\frac{d^2R}{dx^2} = -4$$

Since -4 in less than zero, the function reaches maximum or revenue reaches maximum when x=10.

II - Cost Function

1. If Cost, $C = Q^3 - 3Q^2 + 15Q.\ Find\ the\ Q\ at\ which\ the\ AC\ is\ \min imum\ .$

$$AC = \frac{TC}{Q}$$

$$AC = \frac{Q^3 - 3Q^2 + 15Q}{Q}$$

$$AC = Q^2 - 3Q + 15$$

First order condition for maximization $\frac{d(AC)}{dQ} = 0$

$$\frac{d(AC)}{dQ} = 2Q - 3$$

$$2Q - 3 = 0$$

$$2Q = 3$$

$$Q = \frac{3}{2}; Q = 1.5$$

$$\frac{d^2(AC)}{dQ^2} = 2$$

Hence 2 is greater than zero, the AC reaches minimum.

2. The Cost function is $\frac{1}{3}Q^3 - 4Q^2 + 12Q$. Determine the level of Q where the AC is minimum.

$$AC = \frac{TC}{Q}$$

$$AC = \frac{\frac{1}{3}Q^3 - 4Q^2 + 12Q}{Q}$$

$$AC = \frac{1}{3}Q^2 - 4Q + 12$$

$$\frac{d(AC)}{dQ} = \frac{2}{3}Q - 4$$

First order condition for maximization $\frac{d(AC)}{dQ} = 0$

$$\frac{2}{3}Q - 4 = 0$$

$$\frac{2}{3}Q = 4$$

$$Q = \frac{4}{2/3}$$

$$Q = \frac{12}{2}; Q = 6$$

According to the second order condition $\frac{d^2(AC)}{dQ^2} < 0$

$$\frac{d^2(AC)}{dQ^2} = \frac{2}{3}$$

Hence, $\frac{2}{3} > 0$ the function reaches minimum when Q=6.

III – Profit Function

Profit function is the difference between the total revenue and cost function.

Demand function is $Q = 100 - P$ and the cost function $C = \frac{1}{3}Q^3 - 7Q^2 + 111Q + 50$. Find the firm equilibrium price and output. The firm is at equilibrium when profit is maximum.

The unknown must be Q, then it must be intersected as Q=100-P

$P = 100 - Q$

$$R = P \times Q$$
$$R = (100 - Q)Q$$
$$R = 100Q - Q^2$$

$\Pi = R - C$

$$= (100Q - Q^2) - (\frac{1}{3}Q^3 - 7Q^2 + 111Q + 50)$$

$$\Pi = 100Q - Q^2 - \frac{1}{3}Q^3 + 7Q^2 - 111Q - 50$$

$$\Pi = -11Q + 6Q^2 - \frac{1}{3}Q^3 - 50$$

$\frac{d\Pi}{dQ} = -Q^2 + 12Q - 11$ (Change the sign for solution)

$Q^2 - 12Q + 11 = 0$
$(Q-1)\ \ (Q-11) = 0$
$Q = 1, 11$

$when\, Q = 1$

$$\frac{d^2\Pi}{dQ^2} = -2Q + 12$$

$= -2(1) + 12$
$= -2 + 12$
$= 10$

Hence the function reaches minimum, when Q = 1

$$\frac{d^2\Pi}{dQ^2} = -2Q + 12$$

$$= -2(11) + 12$$
$$= -22 + 12$$
$$= -10$$

Hence the function reaches maximum when Q=11 & equilibrium output is 11 units.

$$P = 100 - Q$$
$$P = 100 - 11$$
$$P = 89$$
$$Answer: Q = 11, P = 89$$

Concept of Maxima & Minima, Elasticity and Point of Inflection

Maxima and Minima

Another key concept in Economics is that of optimization, the calculation of maximum point or minimum point.

Let us take a curve. In the curve for a certain values of x, the gradient is positive, for a further section it is negative, and finally it becomes positive once again. If the curve has positive gradient at one point and a negative gradient shortly afterwards, we may infer that somewhere in between, there is a point at which the gradient is zero. That point is a stationary point-maximum point. If the curve has a negative slope at one point and a positive slope shortly afterwards, we may infer that somewhere in between, there is a point at which the slope is zero. That point is a minimum point.

If at a point the slope is changing from positive through zero to positive again as x increases, so this is neither a maximum nor a minimum, the point is a point of inflexion.

A. Maxima

A function f(x) is said to have attained its "Maximum value" or "Maxima" at x = a, if the function stops to increase and begins to decrease at x = a. In other words, $f(x_1)$ is a maximum value of a function 'f', if it is the highest of all its values for values of x in some neighbourhood of A (Fig. 3.1).

B. Minima

A function f(x) is said to have attained its "Minimum value" or "Minima" at x = b, if the function stops to decrease and begins to increase at x = b. In other words, $f(x_2)$ is a minimum value of a function 'f' if it is the lowest of all its values for values of x in some neighbourhood of B (Fig 3.1).

The Maxima and Minima of the function are called the "Extreme Values" of the function.

Maxima And Minima Of One Variable

Let us consider a function Y = f(x). If we plot this function, the function takes the form as given in the Figure 3.1. We consider three points A, B, and C, where $\frac{dY}{dx} = 0$ in each case. That is in all stationary level the derivative is zero.

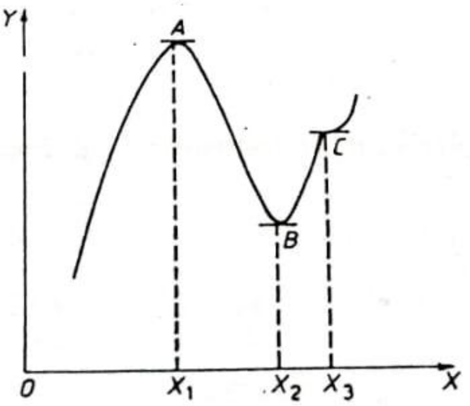


Fig. 3.1 : Maxima and Minima

(1) At Point 'A'

We call the point 'A' a maximum point because Y has a maximum value at this point when X = OX_1. Therefore, we say that at point A,Y is maximum. It means that the value of Y at A is higher than any value on either side of A and also on the left side of point 'A', the curve is increasing and decreasing on the right side of 'A'.This also means that value of Y increases with the increase in X up to the point A. But it must fall after point A has been reached. As value of X increases from X_1 to X_2 the slope of the curve is decreasing or the slope of the curve changes from zero to negative values (since $\frac{dY}{dx} = 0$ at point 'A'). Thus, if 'A' is to be maximum,

i) $\frac{dY}{dx} = 0$ and also

ii) $\frac{d^2Y}{dx^2} < 0$ i.e., –ve at point 'A'.

(2) At Point 'B'

We call the point 'B' a minimum point because Y has a minimum value when X = OX_2. Therefore, we say that at point 'B', Y is minimum. It means that the value of Y at 'B' is lower than any value on either side of B and also on the left side of point 'B', the curve is decrcasing and increasing on the right side of 'B'. This also means that value of Y decreases with the increase in X upto the point 'B'. But, it must increase after point 'B' has been reached. As value of X increases form X_2 to X_3, the slope of the curve is increasing or the slope of the curve changes from zero to positive values (since $\frac{dY}{dx} = 0$ at point 'B').

Thus if 'B' is to be minimum,

i) $\frac{dY}{dx} = 0$ and also

ii) $\frac{d^2Y}{dx^2} > 0$ i.e., ; +ve at point 'B'.

Table 3.2 : Conditions for Maxima and Minima

	Maxima	Minima
1. First Order Condition (Necessary Condition)	f ' (x) or $\frac{dY}{dx} = 0$	f ' (x) or $\frac{dY}{dx} = 0$
2. Second Order Condition (Sufficient Condition)	f '' (x) or $\frac{d^2Y}{dx^2} < 0$ or –ve	f '' (x) or $\frac{d^2Y}{dx^2} > 0$ or +ve

(3) At Point 'C'

Point of Inflexion is a point at which a curve is changing from concave upward to concave downward, or vice versa. In the Figure 3.1,we call this point 'C' a"Point of Inflection" or "Inflexional Point". Because on either sides of point 'C' the curve slopes upwards. Therefore, on either sides of 'C', the first order derivative is greater than zero i.e., positive except at point C. Hence, this point is called "Inflexional Point" or "Point of Inflexion", because of mere bend in the curve. At the inflexional points, the second order derivative is equal to zero. "Inflexional points may be stationary and inflexional. In this case, both first and second order derivatives are zero. If the point is inflexional and non statioary, the first order derivative is not equal to zero. But the second order derivative is equal to zero.

Thus, if 'C' is to be "Inflexional point", then

i) $\frac{dY}{dx} \geq 0$ and also ii) $\frac{d^2Y}{dx^2} = 0.$

Examples:

1. Given the function $y = x^3 - 3x^2 + 7$, find the point of inflexion.

Solution:

$$\frac{dy}{dx} = 3x^2 - 6x$$

The first condition for inflexion is $\frac{dy}{dx} = 0$

Therefore, $3x^2 - 6x = 0$

$x(3x - 6) = 0$

i) $x = 0$ or ii) $3x - 6 = 0$

$3x = 6$

$x = \frac{6}{3} = 2.$

Thus, $\frac{dy}{dx} = 0$, when $x = 0$ or $x = 2$.

a) If $x = 0$, $y = (0)^3 - 3(0)^2 + 7$

$y = 7$

b) If $x = 2$, $y = (2)^3 - 3(2)^2 + 7$

$y = 8 - 12 + 7 = 3$

The second condition for inflexion is $\frac{d^2y}{dx^2} = 0$

$$\frac{d^2y}{dx^2} = 6x - 6$$

$6x - 6 = 0$

$6x = 6$

$x = \frac{6}{6} = 1 > 0.$

The point of inflexion is $x = 0$ and $y = 7$ or $x = 2$ and $y = 3$.

Find the maxima and minima of the function $y = 2x^3 - 6x$.

Solution:

$y = 2x^3 - 6x$

$$\frac{dy}{dx} = 6x^2 - 6$$

At the maximum or minimum $\frac{dy}{dx} = 0$

Therefore, $6x^2 - 6 = 0$

$6x^2 = 6$

$x^2 = \frac{6}{6}$

$x^2 = 1$

$x = \pm 1$

$x = -1$ and $x = 1$ give maximum or minimum

$$\frac{dy}{dx} = 6x^2 - 6x$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

When $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$ i.e., negative

When $x = 1$, $\frac{d^2y}{dx^2} = 6 > 0$ i.e., positive.

Therefore, $x = -1$ gives the maximum value of the function and $x = 1$ gives the minimum value of the function.

Profit maximization

Profit maximization is the process by which a company determines the price and production level that returns the greatest profit. This concept is a key goal in traditional economic theories, especially within the realm of microeconomics. Here are some key points:

Key Concepts

1. **Total Revenue (TR) and Total Cost (TC):**
   - Total Revenue is the total income a company generates from selling its products or services (Price x Quantity).
   - Total Cost includes all the expenses incurred in the production process (fixed and variable costs).

2. **Profit:**
   - Profit is the difference between Total Revenue (TR) and Total Cost (TC).
   - Profit=Total Revenue−Total Cost\ {Profit} = {Total Revenue} - {Total Cost}Profit=Total Revenue−Total Cost

3. **Marginal Revenue (MR) and Marginal Cost (MC):**

- Marginal Revenue is the additional revenue gained from selling one more unit of a product.
- Marginal Cost is the additional cost incurred from producing one more unit of a product.
- The profit-maximizing output level is where Marginal Revenue (MR) equals Marginal Cost (MC): MR=MC

Cost minimization

Cost minimization is a strategy or approach used by businesses and organizations to reduce expenses and optimize resources while maintaining productivity and quality. It involves identifying areas where costs can be cut or minimized without compromising the efficiency or effectiveness of operations. This can include measures such as:

1. **Streamlining Processes:** Identifying and eliminating unnecessary steps or inefficiencies in workflows and operations.
2. **Negotiating with Suppliers:** Negotiating better deals with suppliers for raw materials, equipment, or services to lower procurement costs.
3. **Optimizing Inventory:** Managing inventory levels efficiently to reduce holding costs while ensuring adequate stock to meet demand.
4. **Utilizing Technology:** Implementing technology solutions such as automation, digital tools, and software systems to improve efficiency and reduce manual labor costs.
5. **Energy Efficiency:** Implementing energy-saving measures to reduce utility costs, such as using energy-efficient equipment and optimizing lighting and heating/cooling systems.
6. **Outsourcing and Offshoring:** Exploring outsourcing or offshoring options for non-core functions to lower labor costs while maintaining quality standards.
7. **Training and Development:** Investing in training programs to enhance employee skills and productivity, reducing errors and rework costs.
8. **Monitoring and Analysis:** Regularly monitoring expenses, analyzing cost drivers, and identifying opportunities for further cost reduction.

9. **Benchmarking:** Comparing costs and performance metrics with industry peers or best practices to identify areas for improvement.

Cost minimization is a continuous process that requires ongoing evaluation, adaptation, and decision-making to achieve sustainable cost reductions while supporting organizational goals and objectives.

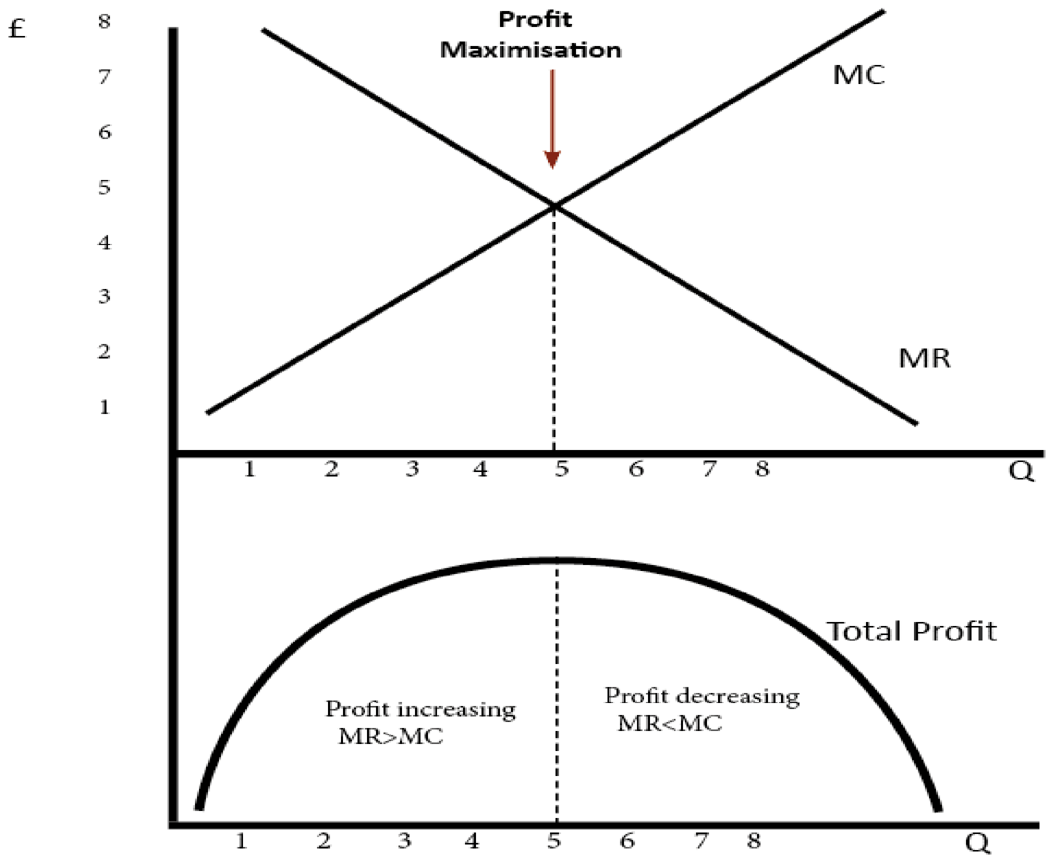
Profit & revenue maximization under perfect competition under monopoly

An assumption in classical economics is that firms seek to maximise profits.

Profit = Total Revenue (TR) – Total Costs (TC).

Therefore, profit maximisation occurs at the biggest gap between total revenue and total costs.

A firm can maximise profits if it produces at an output where marginal revenue (MR) = marginal cost (MC)



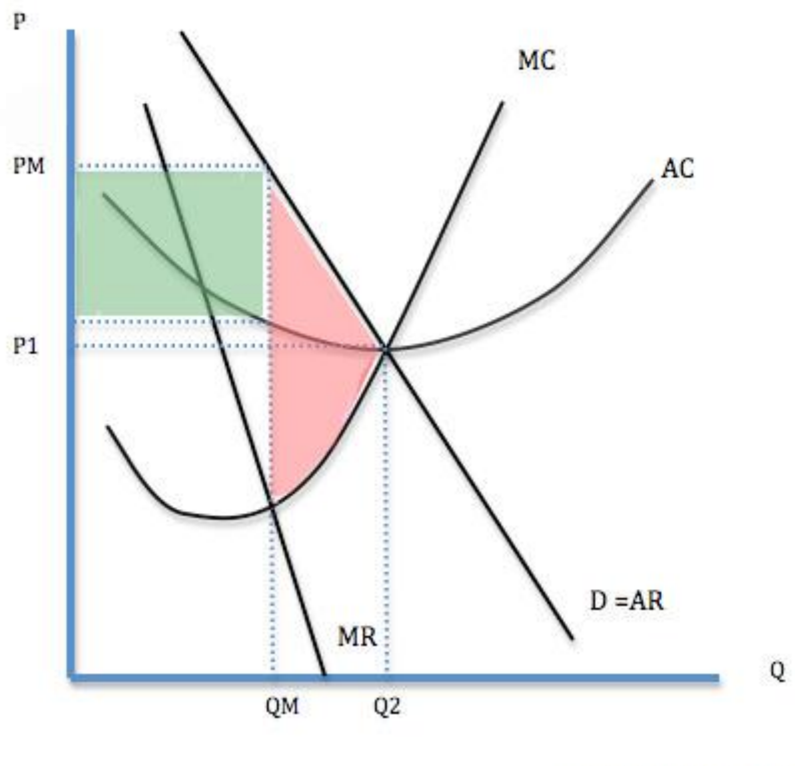
To understand this principle look at the above diagram.

If the firm produces less than Output of 5, MR is greater than MC. Therefore, for this extra output, the firm is gaining more revenue than it is paying in costs, and total profit will increase.

At an output of 4, MR is only just greater than MC; therefore, there is only a small increase in profit, but profit is still rising.

However, after the output of 5, the marginal cost of the output is greater than the marginal revenue. This means the firm will see a fall in its profit level because the cost of these extra units is greater than revenue.

Profit maximization under monopoly

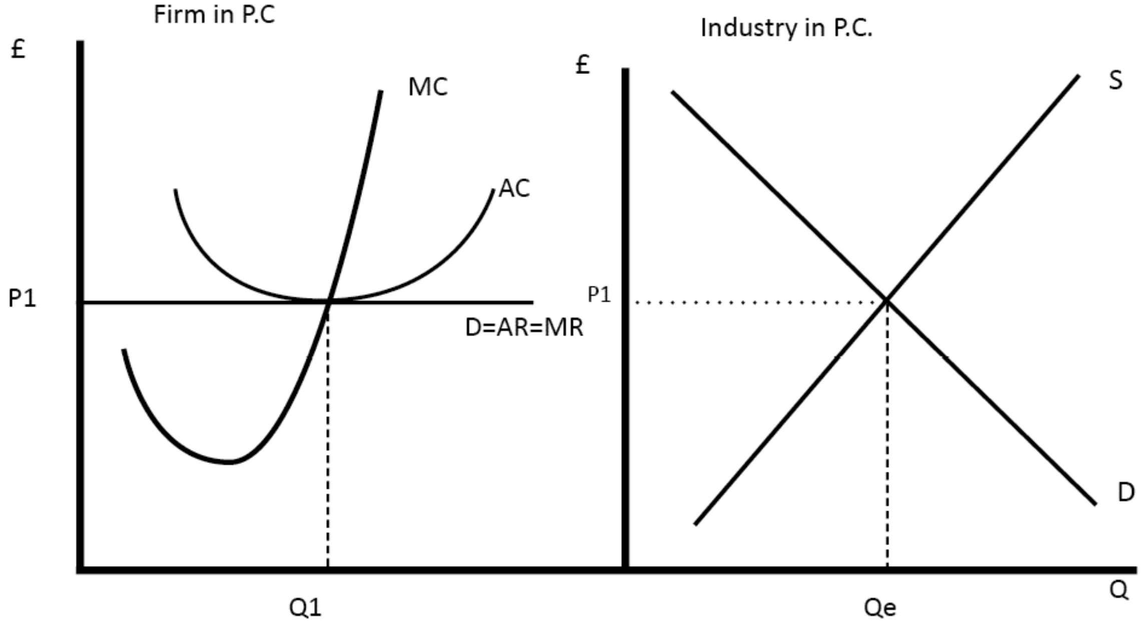


In this diagram, the monopoly maximises profit where MR=MC – at Qm. This enables the firm to make supernormal profits (green area). Note, the firm could produce more and still make normal profit. But, to maximise profit, it involves setting a higher price and lower quantity than a competitive market.

Note, the firm could produce more and still make a normal profit. But, to maximise profit, it involves setting a higher price and lower quantity than a competitive market.

Therefore, in a monopoly profit maximisation involves selling a lower quantity and at a higher price.

Profit maximization under Perfect Competition



In perfect competition, the same rule for profit maximisation still applies. The firm maximises profit where MR=MC (at Q1).

For a firm in perfect competition, demand is perfectly elastic, therefore MR=AR=D.

This gives a firm normal profit because at Q1, AR=AC.

Maximizing excise tax revenue in monopolistic competitive market

In a monopolistically competitive market, the rule for maximizing profit is to set MR = MC—and price is higher than marginal revenue, not equal to it because the demand curve is downward sloping.

Profit is maximized where marginal revenue is equal to marginal cost. In this case, for a competitive firm, marginal revenue is equal to price. So profit is maximized where price is equal to marginal cost or at this point right here.

Minimization of cost

Cost minimisation is a financial strategy that aims to achieve the most cost-effective way of delivering goods and services to the require level of quality. It is important to remember that cost minimisation is not about reducing quality or short-changing customers – it always remains important to meet customer needs.

Cost Minimisation for a Given Output:

In the theory of production, the profit maximisation firm is in equilibrium when, given the cost- price function, it maximises its profits on the basis of the least cost combination of factors. For this, it will choose that combination which minimises its cost of production for a given output. This will be the optimal combination for it.

Assumptions:

This analysis is based on the following assumptions:

1. There are two factors, labour and capital.
2. All units of labour and capital are homogeneous.
3. The prices of units of labour (w) and that of capital (r) are given and constant.
4. The cost outlay is given.
5. The firm produces a single product.
6. The price of the product is given and constant.
7. The firm aims at profit maximisation.
8. There is perfect competition in the factor market.

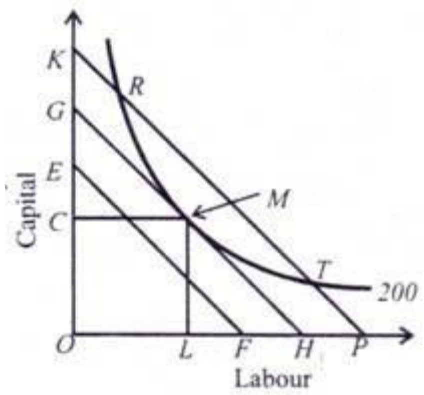


Fig. 15

Explanation:

Given these assumptions, the point of least-cost combination of factors for a given level of output is where the isoquant curve is tangent to an isocost line. In Figure 15, the isocost line GH is tangent to the isoquant 200 at point M. The firm employs the combination of OC of capital and OL of labour to produce 200 units of output at point M with the given cost-outlay GH.

At this point, the firm is minimising its cost for producing 200 units. Any other combination on the isoquant 200, such as R or T, is on the higher isocost line KP which shows higher cost of production. The isocost line EF shows lower cost but output 200 cannot be attained with it. Therefore, the firm will choose the minimum cost point M which is the least-cost factor combination for producing 200 units of output. M is thus the optimal combination for the firm.

The point of tangency between the isocost line and the isoquant is an important first order condition but not a necessary condition for the producer's equilibrium.

There are two essential K or second order conditions for the equilibrium of the firm:

1. The first condition is that the slope of the isocost line must equal the slope of the isoquant curve. The slope of the isocost line is equal to the ratio of the price of labour (w) and the price of capital (r). The slope of the isoquant curve is equal to the marginal rate of technical substitution of labour and capital (MRTSLK) which is, in turn, equal to the

ratio of the marginal product of labour to the marginal product of capital (MPL/MPK' condition for optimality can be written as.

w/r MPL/MPK = MRTSLK

The second condition is that at the point of tangency, the isoquant curve must be convex to the origin. In other words, the marginal rate of technical substitution of labour for capital (MRTSLK) must be diminishing at the point of tangency for equilibrium to be stable. In Figure 16, S cannot be the point of equilibrium for the isoquant IQ1 is concave where it is tangent to the isocost line GH. At point S, the marginal rate of technical substitution between the two factors increases if move to the right or left on the curve IQ1.

Moreover, the same output level can be produced at a lower cost AB or EF and there will be a comer solution either at C or F. If it decides to produce at EF cost, it can produce the entire output with only OF labour. If, on the other hand, it decides to produce at a still lower cost CD, the entire output can be produced with only OC capital.

Both the situations are impossibilities because nothing can be produced either with only labour or only capital. Therefore, the firm can produce the same level of output at point M, where the isoquant curve IQ is convex to the origin and is tangent to the isocost line GH. The analysis assumes that both the isoquants represent equal level of output, IQ = IQ1.

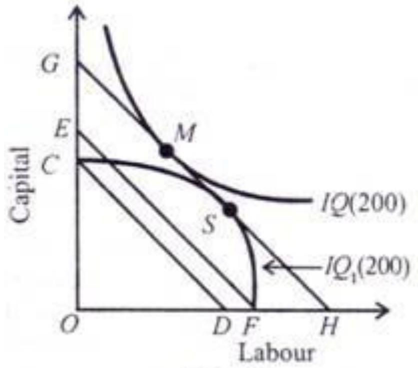


Fig. 16

Output-Maximisation for a Given Cost:

The firm also maximises its profits by maximising its output, given its cost outlay and the prices of the two factors. This analysis is based on the same assumptions, as given above. The conditions for the equilibrium of the firm are the same, as discussed above.

1. The firm is in equilibrium at point P where the isoquant curve 200 is tangent to the isocost line CL in Figure 17. At this point, the firm is maximising its output level of 200 units by employing the optimal combination of OM of capital and ON of labour, given its cost outlay CL.

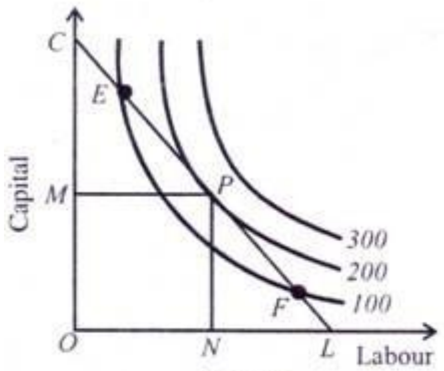


Fig. 17

But it cannot be at points E or F on the isocost line CL, since both points give a smaller quantity of output, being on the isoquant 100, than on the isoquant 200. The firm can reach the optimal factor com-bination level of maximum output by moving along the isocost line CL from either point E or F to point P.

This movement involves no extra cost because the firm re-mains on the same isocost line. The firm cannot attain a higher level of output such as isoquant 300 because of the cost constraint. Thus the equilibrium point has to be P with optimal factor combination OM+ ON. At point P, the slope of the isoquant curve 200 is equal to the slope of the isocost line CL. It implies w/r = MPL/MPK = MRTSLK.

2. The second condition is that the isoquant curve must be convex to the origin at the point of tangency with the isocost line, as explained above in terms of Figure 16.